American Options

The value of an American option can be found recursively by working from right to left:

$$V(S_t, K, t) = Max\left\{ \cancel{ExerciseValue}, \; e^{-rh}\left[(p^*)V(S_t u, K, t+h) + (1-p^*)V(S_t d, K, t+h)\right]\right\}$$

Payoff$_t$ = G_t = intrinsic value at t

where:

$$ExerciseValue = \begin{cases} S_t - K_t & \text{if the option is a call option} \\ K_t - S_t & \text{if the option is a put option} \end{cases}$$

$V(S_t, K, t) =$ Value of an option at time t on an underlying asset having a price of S_t at time t

Suppose that an American put option has a strike price of \$50. The current stock price is \$52, and the volatility is 30%. The stock follows the binomial model with each period being 0.25 years in length. The stock does not pay dividends, and the risk-free rate of return is 10%.

As in the previous section, we have:

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{(0.10-0.00)(0.25)+0.30\sqrt{0.25}} = 1.19125$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{(0.10-0.00)(0.25)-0.30\sqrt{0.25}} = 0.88250$$

$$p^* = \frac{1}{e^{\sigma\sqrt{h}} + 1} = \frac{1}{e^{0.30\sqrt{0.25}} + 1} = 0.4626$$

The stock price tree is the same as in the two previous sections:

			87.904
		73.792	
	61.945		65.121
52.000		54.666	
	45.890		48.243
		40.498	
			35.739

The American put option price tree is constructed by working from right to left:

			0.000
		0.000	
	0.483		0.000
3.046		0.921	
	5.396		1.757
		9.502	
			14.261

The rightmost column of the option price tree is easily found as:

$$Max\left[0, \; 50 - S_{0.75}\right]$$

The values in the nodes just to the left of the rightmost column are:

$$0.000 = Max\left\{50 - 73.792,\ e^{-0.10(0.25)}\left[(0.4626)\times 0.000 + (1-0.4626)\times 0.000\right]\right\}$$

$$0.921 = Max\left\{50 - 54.666,\ e^{-0.10(0.25)}\left[(0.4626)\times 0.000 + (1-0.4626)\times 1.757\right]\right\}$$

$$9.502 = Max\left\{50 - 40.498,\ e^{-0.10(0.25)}\left[(0.4626)\times 1.757 + (1-0.4626)\times 14.261\right]\right\}$$

The value of 9.502 is bolded in the tree above to indicate that early exercise is optimal at that node.

Working from right to left, we can find the two possible prices after 0.25 years:

$$0.483 = Max\left\{50 - 61.945,\ e^{-0.10(0.25)}\left[(0.4626)\times 0.000 + (1-0.4626)\times 0.921\right]\right\}$$

$$5.396 = Max\left\{50 - 45.890,\ e^{-0.10(0.25)}\left[(0.4626)\times 0.921 + (1-0.4626)\times 9.502\right]\right\}$$

We can now find the initial price of the American put option:

$$3.046 = Max\left\{50 - 52,\ e^{-0.10(0.25)}\left[(0.4626)\times 0.483 + (1-0.4626)\times 5.396\right]\right\}$$

We cannot use the direct method to find the value of an American option.

◊◊◊ 4.3 Options on Futures Contracts

Futures contracts are similar to forward contracts in that both allow a buyer and a seller to lock in a price now for a transaction scheduled to occur in the future. The locked-in price is known as the futures price. There is no initial cost to enter into a futures contract.

Entering into an agreement to buy the underlying asset in the future using a futures contract is often described as "buying the futures contract" or "going long the futures contract." Entering into an agreement to sell the underlying asset in the future using a futures contract is often described as "selling the futures contract" or "going short the futures contract."

In this section, we assume that futures prices are equal to forward prices, and we use the same notation for both. Therefore a futures price at time t for a contract that expires at time T_F can be written as:

$$F_{t,T_F} = S_t e^{(r-\delta)(T_F - t)}$$

A call option on a futures contract has a payoff that increases with the futures price, and a put option on a futures contract has a payoff that increases as the futures price falls. If the option is exercised at time T, the payoff is:

Payoff for a call option on a futures contract $= Max(0, F_{T,T_F} - K)$

Payoff for a put option on a futures contract $= Max(0, K - F_{T,T_F})$